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Evanescent couplings are not renormalizable

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Abstract. Evanescent interactions like $\bar{\psi}\{\overleftrightarrow{\partial}, \Gamma_{(KLMN)}\}\psi\phi^{KLMN}$, which disappear in four dimensions but which lead to divergences on the basis of power counting, are shown to be non-renormalizable. This result is not apparent at the one-loop level when the S-matrix elements are simple polynomials in the four-dimensional limit.

1. Introduction

If taken seriously, the technique of dimensional regularization (t'Hooft and Veltman 1972, Bollini and Giambiagi 1972) compels us to view field theories in arbitrary ($2l$) dimensions before one proceeds to the four-dimensional limit $l = 2$. In the course of these studies one is led to 'anomalous currents' (Akyeampong and Delbourgo 1973a, b, 1974) such as $\bar{\psi}\{\overleftrightarrow{\partial}, \Gamma_{(KLMN)}\}\psi$ in axial current Ward identities, currents which disappear for $l = 2$, but whose matrix elements yield the Adler anomalies. Interactions which fade away in four dimensions (or, stronger still, cannot even be written down!) have been coined 'evanescent' by Bollini and Giambiagi (1974). It is the interplay of their vanishing and the divergence of Feynman integrals for $l \to 2$ which is responsible for the interesting finite corrections to classical Ward identities.

In this paper we shall investigate a theory which has a primary evanescent interaction (unlike Bollini and Giambiagi (1974) we shall adhere to purely local field couplings in the Lagrangian itself) namely, $\mathscr{L}_1 = G\bar{\psi}\{\overleftrightarrow{\partial}, \Gamma_{KLMN}\}\phi^{KLMN}$, where ϕ stands for the 'pseudoscalar' field in n dimensions. On the basis of power counting $\mathscr{L}_1$ is singular as x^{-5} near $l = 2$ (signalled by G having dimensions M^{-1}) and one would naturally argue that the divergences get worse in higher orders of perturbation theory causing the model to be non-renormalizable. However, the interaction itself disappears at $l = 2$, so the question arises whether the theory is really infinite at all and if renormalizability is truly lost. We shall prove that the model is indeed non-renormalizable, but to arrive at this conclusion we will need to go beyond the one-loop level, ie the bad effects are at least of order $\hbar^2$. The basic reason is as follows: at the one-loop level the divergent Feynman integrals contain a pole term $(l-2)^{-1}$ and these multiply a factor $(l-2)$ which must be present for all form factors associated with kinematic terms that survive the four-dimensional limit (because $\mathscr{L}_1 \to 0$ as $l \to 2$). The product of these yields a polynomial in external momenta and masses in four dimensions. At the next, two-loop, level we may encounter double integrals which contain *second*-order poles $(l-2)^{-2}$ but only a *single* factor $(l-2)$ in the numerator, signifying a divergence. (Another way of stating this is to note that the one-loop polynomial suffers a further divergent integration with no further compensating zero from $\mathscr{L}_1$.) As these divergences get progressively worse in higher orders of G there is no hope of renormalizing the theory.

The final result, that evanescent couplings with bad power counting characteristics are non-renormalizable after all, is useful in restricting the class of Lagrangian models that are viable in the context of dimensional regularization.

Let us now substantiate these statements by giving a few details of our investigation. For our free Lagrangian we shall take a massive fermion ψ and a massless boson ϕ:

$$\mathscr{L}_0 = -\phi^{+KLMN}\partial^2\phi_{KLMN} + \bar{\psi}(\mathrm{i}\not{\partial} - m)\psi$$

in order to simplify some of the Feynman integrals without affecting the ultraviolet behaviours in question. The fact that $\mathscr{L}_0$ can lead to ghost mesons in some of the ϕ components will not concern us unduly, since none arises when $l \to 2$. The classical tree graphs evidently give zero identically in the four-dimensional limit, so the first interesting results occur at the one-loop level. In momentum space the vertex factor arising from $\mathscr{L}_1$ in a perturbation development is $\{2\not{p}+\not{k}, \Gamma_{KLMN}\}$ where p and $p+k$ stand for the incoming and outgoing fermion momenta. These have to be combined with the propagators $S(p) = \mathrm{i}(\not{p}-m)^{-1}$ and $D^{N_1\ldots N_4}_{M_1\ldots M_4}(k) = \mathrm{i}\delta^{N_1\ldots N_4}_{M_1\ldots M_4}/k^2$ by the standard Feynman rules. As we shall be interested in kinematic terms produced from Feynman graphs which survive the passage to four dimensions one can set *external* momenta equal to *zero* at each vertex.

We may now determine some simple one-loop diagrams.

2. Boson self-energy

To order G^2, retaining the part which survives four dimensions,

$$\Pi^{(N_1\ldots N_4)}_{(M_1\ldots M_4)}(k) = 4\mathrm{i}G^2\int\frac{\mathrm{d}^{2l}p}{(2\pi)^{2l}}\frac{\mathrm{Tr}[\{\not{p},\Gamma_{M_1\ldots M_4}\}(\not{p}+m)\{\not{p},\Gamma^{N_1\ldots N_4}\}(\not{p}+\not{k}+m)]}{(p^2-m^2)[(p+k)^2-m^2]}.$$

Introducing a Feynman parameter α, shifting the integral, and dropping all $\{\not{k},\Gamma\}$ terms, the usual manipulations lead us to

$$\begin{aligned}\Pi^{(N_1\ldots N_4)}_{(M_1\ldots M_4)}(k) &= \frac{G^2}{(2\pi)^l}\Gamma(3-l)\delta^{(N_1\ldots N_4)}_{(M_1\ldots M_4)}\frac{16(2l+1)}{l(l-1)}\int[m^2-k^2\alpha(1-\alpha)]^l\,\mathrm{d}\alpha \\ &\xrightarrow[l\to 2]{} G^2(k^4-10m^2k^2+30m^4)\delta^{(N_1\ldots N_4)}_{(M_1\ldots M_4)}/3\pi^2. \qquad (1)\end{aligned}$$

As promised the quartic divergence has disappeared owing to the vanishing trace.

3. Fermion self-energy

Since we shall presently take this graph to be part of a larger graph the integral to be evaluated is

$$\Sigma(p) = -\mathrm{i}g^2\int\frac{\mathrm{d}^{2l}k}{(2\pi)^{2l}}\frac{\{2\not{p}+\not{k},\Gamma_{M_1\ldots M_4}\}(\not{p}+\not{k}+m)\{2\not{p}+\not{k},\Gamma^{M_1\ldots M_4}\}}{[(p+k)^2-m^2]k^2}.$$

The calculation of the numerator here (as well as that of the vertex part to follow) is greatly facilitated by the methods set out in an appendix. Using the lemmas worked out there the final answer is a polynomial in $\not{p}$:

$$\Sigma(p) \sim G^2\Gamma(3-l)(2l-1)(2l-3)\int_0^1 \mathrm{d}\alpha(\{\not{p}[3(l-1)-\alpha(1+l)]+ml\}[p^2\alpha(1-\alpha)-\alpha m^2]^{l-1}$$

$$-(l-1)p^2(2-\alpha)^2[(1-\alpha)\not{p}+m][p^2\alpha(1-\alpha)-\alpha m^2]^{l-2})$$

$$\xrightarrow[l\to 2]{} G^2[(\not{p}+2m)(p^2+\tfrac{1}{2}m^2)+\tfrac{1}{6}\not{p}p^2]. \tag{2}$$

4. Vertex part

$$V_{M_1\ldots M_4}(p',p) = -\mathrm{i}G^3\int\frac{\mathrm{d}^{2l}k}{(2\pi)^{2l}}\frac{\{\not{k},\Gamma_{N_1\ldots N_4}\}(\not{p}+\not{k}+m)\{2\not{k},\Gamma_{M_1\ldots M_4}\}(\not{p}+\not{k}+m)\{\not{k},\Gamma^{N_1\ldots N_4}\}}{[(p'+k)^2-m^2]k^2[(p+k)^2-m^2]}$$

if we are only interested in kinematic terms which survive $l\to 2$. Combining denominators with Feynman parameters and using simplification methods outlined in the appendix we end up with

$$V_{M_1\ldots M_4}(p',p) = X\Gamma_{M_1\ldots M_4}+Y[\not{p}'-\not{p},\Gamma_{M_1\ldots M_4}] \tag{3}$$

where the form factors X and Y tend to

$$X \to Y/405m \propto G^3(p^2+p'^2-4m^2) \tag{3'}$$

for four dimensions.

5. Other one-loop graphs

The systematics should by now be obvious. Every one-loop graph is divergent due to the momentum factor in the vertex, but this is cancelled by a zero caused by the disappearance of $\mathscr{L}_1$. Always we are left with a polynomial in external momenta whose degree increases with G—as it must from simple dimensional analysis.

We may now wonder if this phenomenon carries over to higher loops and if all Feynman diagrams are finite. The answer is 'no' and is most simply illustrated by examination of the vacuum graphs. The simplest two-loop graph (conveniently treated in x space) does indeed happen to be finite, but this is only an accident due to the masslessness of our boson. Thus

$$Z = G^2\int \mathrm{d}^{2l}x\ \mathrm{Tr}(\{\vec{\partial},\Gamma_{M_1\ldots M_4}\}S(x)\{\vec{\partial},\Gamma^{M_1\ldots M_4}\}S(x))D(x).$$

Retaining the most singular terms in the integrand and dropping all $\partial^2 D$ terms upon rotation to Euclidean space,

$$Z \sim G^2\int \mathrm{d}^{2l}x 2^l D\partial^M\partial^N D\partial_M\partial_N D(l-2)$$

$$= G^2\int \mathrm{d}^{2l}x 2^l 8l(2l-1)(l-1)^2(l-2)D^3/x^4$$

$$\rightarrow 192(l-2)G^2 \int d^{2l}x D^3/x^4$$

$$\sim (l-2)G^2 \int_0^{\infty} (r^2)^{l-6}\, dr^2$$

which is ultraviolet-finite as $l \rightarrow 2$. Another way of seeing this is to work in momentum space, carry out the fermion loop integral, obtaining Π (a polynomial at $l = 2$), and then working out the integral

$$\int d^{2l}k \Pi^{M_1 \dots M_4}_{M_1 \dots M_4}(k)/k^2.$$

In dimensional regularization this is zero for polynomial Π. However, had we chosen to give the mesons a mass μ we would instead have obtained the divergent answer $G^2(\mu^2)^{2l-1}/(l-2)$.

The problem is much clearer at the three-loop level. Consider first the vacuum diagram of figure 1. If we carry out the meson loop integrations first we are left with

$$Z \sim G^2 \int d^{2l}p \operatorname{Tr}[\Sigma(p)(\not{p}-m)^{-1}\Sigma(p)(\not{p}-m)^{-1}]$$

and since $\Sigma(p)$ is finite near $l = 2$ (see equation (2)) this final integral is bound to diverge near four dimensions. A more relevant example is the meson–meson scattering diagram of figure 2. Carrying out the fermion loop integrations first

$$M = \int d^{2l}k_5\, d^{2l}k_6 \delta(k_5+k_6-k_1-k_2)M_4(k_1k_2, k_5k_6)M_4(k_5k_6, k_3k_4)/k_5^2k_6^2.$$

But M_4 is a finite polynomial in k near $l = 2$. Hence the final integration produces a pole term $(l-2)^{-1}$.

The inescapable conclusion then is that the higher-loop graphs diverge in general. Because these diagrams are associated with higher powers of G they require ever increasing numbers of subtractions and the theory is therefore non-renormalizable. We can

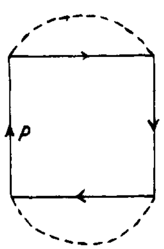

Figure 1. A divergent three-loop vacuum graph.

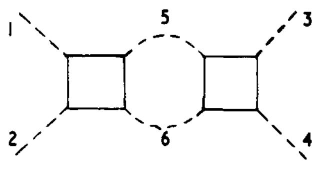

Figure 2. A divergent three-loop meson scattering graph.

thus class all theories with evanescent interactions and coupling constants having dimensions of inverse mass powers as undesirable in spite of appearances. More importantly, this means that if we start with a renormalizable theory and happen to meet anomalous currents in the context of Ward identities, we should never attempt to cancel them off with evanescent counter Lagrangians.

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Appendix

Let $\Gamma_{(r)}$ be a shorthand for $\Gamma_{[M_1 \ldots M_r]}$, the antisymmetric product of r Γ matrices. In the text we meet numerators of Feynman integrals of the type $\not{k}\Gamma_{(4)}\not{p}\Gamma'_{(4)}\not{k}\Gamma^{(4)}$ where k is an internal and p is an external momentum. To simplify expressions like these we continually apply the following formulae (Delbourgo and Prasad 1974a, b):

$$[\Gamma_{(1)}, \Gamma_{(r)}] = \begin{cases} \Gamma_{(r-1)} & \text{for } r \text{ even} \\ \Gamma_{(r+1)} & \text{for } r \text{ odd} \end{cases}$$

$$\{\Gamma_{(1)}, \Gamma_{(r)}\} = \begin{cases} \Gamma_{(r+1)} & \text{for } r \text{ even} \\ \Gamma_{(r-1)} & \text{for } r \text{ odd} \end{cases}$$

$$C(s, r)\Gamma_{(r)} = 2^{-l}\Gamma_{(s)}\Gamma_{(r)}\Gamma^{(s)},$$

where

$$C(s, r) = 2^{-l}\sum_q (-1)^{sr+q}\binom{2l-r}{s-q}\binom{r}{q}$$

is the sr element of the Fierz transformation in $2l$ dimensions.

The method is best illustrated by working out two examples:

$$\begin{aligned}
\not{k}\Gamma_{(4)}\Gamma'_{(4)}\not{k}\Gamma^{(4)} &\to (2l)^{-1}k^2\Gamma_{(1)}\Gamma_{(4)}\Gamma'_{(4)}\Gamma^{(1)}\Gamma^{(4)} \\
&= (4l)^{-1}k^2\Gamma_{(1)}\Gamma_{(4)}([\Gamma'_{(4)}, \Gamma^{(1)}] + \{\Gamma'_{(4)}, \Gamma^{(1)}\})\Gamma^{(4)} \\
&= 2^l(4l)^{-1}k^2\Gamma_{(1)}(C(4,3)[\Gamma'_{(4)}, \Gamma^{(1)}] + C(4,5)\{\Gamma'_{(4)}, \Gamma^{(1)}\}) \\
&= 2^{2l}(4l)^{-1}k^2\Gamma'_{(4)}[C(4,3)\{C(1,4) - C(1,0)\} + C(4,5)\{C(1,4) + C(1,0)\}]
\end{aligned}$$

$$\begin{aligned}
\Gamma_{(1)}\Gamma_{(4)}\not{p}\Gamma'_{(4)}\Gamma^{(1)}\Gamma^{(4)} &= 2^{l-2}\Gamma_{(1)}\Gamma_{(4)}(\{\{\not{p}, \Gamma'_{(4)}\}, \Gamma^{(1)}\} + [\{\not{p}, \Gamma'_{(4)}\}, \Gamma^{(1)}] + \{[\not{p}, \Gamma'_{(4)}], \Gamma^{(1)}\} \\
&\quad + [[\not{p}, \Gamma'_{(4)}], \Gamma^{(1)}])\Gamma^{(4)} \\
&\to 2^{l-2}\Gamma_{(1)}(C(4,2)\{[\not{p}, \Gamma_{(4)}], \Gamma^{(1)}\} + C(4,4)[[\not{p}, \Gamma'_{(4)}], \Gamma^{(1)}]) \\
&= 2^{2l-2}[\not{p}, \Gamma'_{(4)}](C(4,2)\{C(1,3) + C(1,0)\} + C(4,4)\{C(1,3) - C(1,0)\})
\end{aligned}$$

when we discard the kinematic term $\{\not{p}, \Gamma'_{(4)}\}$ which has no place in four dimensions. Other Feynman integral numerators can be simplified in much the same way.

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